(d) $x \ln x = y(1+\sqrt{3+y^2})y'$, $y(1) = 1$

$$y\left(1+\sqrt{3+y^2}\right)\frac{dy}{dx} = x \ln x \qquad \text{(separável)}$$

$$\Rightarrow \int y\left(1+\sqrt{3+y^2}\right) dy = \int x \ln x \, dx$$

- $\int x \ln x \, dx \qquad \left(\begin{array}{lcl} u = \ln x & \Rightarrow & du = \frac{1}{x}\,dx \\ dv = x\,dx & & v = \frac{x^2}{2} \end{array}\right)$

$$= \frac{x^2}{2}\cdot \ln x - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C_1 = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C_1$$

- $\int y\left(1+\sqrt{3+y^2}\right) dy = \int \left(1+\sqrt{3+y^2}\right) y\,dy \qquad \left(\begin{array}{l} u = 3+y^2 \\ du = 2y\,dy \end{array}\right)$

$$= \int \left(1+\sqrt{u}\right)\frac{1}{2}\,du = \frac{1}{2}\int 1 + u^{1/2}\,du = \frac{1}{2}\left(u + \frac{2}{3}u^{3/2}\right) + C_2$$

$$= \frac{1}{2}\left[3+y^2 + \frac{2}{3}\left(3+y^2\right)^{3/2}\right] + C_2$$

Resolva a equação diferencial $y' = x + y$ utilizando a mudança de variáveis $u = x + y$.

$u(x) = x + y(x)$

$$u = x + y \Rightarrow y = u - x \Rightarrow y' = (u-x)' = u' - 1$$

$$\therefore \; y' = x + y \Rightarrow u' - 1 = u \Rightarrow u' = u + 1$$

$$\Rightarrow \frac{1}{u+1} \cdot u' = 1 \quad \text{(separável)}$$

$$\Rightarrow \int \frac{1}{u+1}\, du = \int 1\, dx \Rightarrow \ln|u+1| = x + c$$

$$\Rightarrow |u+1| = e^{x+c} \Rightarrow \begin{cases} u+1 = e^{x+c}, & u+1 \geq 0 \\ -(u+1) = e^{x+c}, & u+1 < 0 \end{cases}$$

$$\Rightarrow \begin{cases} u+1 = e^{x} \cdot k_1 \\ -u-1 = e^{x} \cdot k_2 \end{cases} \Rightarrow \begin{cases} u = k_1 e^{x} - 1 \\ u = -k_2 e^{x} - 1 \end{cases} \Rightarrow u = k e^{x} - 1, \; k \neq 0.$$

Portanto,

$$x + y = k e^{x} - 1 \Rightarrow y = -x - 1 + k e^{x}$$

$y' = x + y \Rightarrow y' - y = x$ é linear

Fator integrante: $e^{\int -1 dx} = e^{-x}$

$\therefore e^{-x} \cdot (y' - y) = e^{-x} \cdot x \Rightarrow e^{-x} \cdot y' - e^{-x} y = x e^{-x}$

$\Rightarrow (e^{-x} \cdot y)' = x e^{-x} \Rightarrow \int (e^{-x} y)' dx = \int x e^{-x} dx$

Resolvendo a integral do segundo membro por partes:

$u = x$ $\quad$ $du = dx$

$dv = e^{-x} dx \Rightarrow v = -e^{-x}$

$\therefore \int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C$

Assim,

$e^{-x} y = -x e^{-x} - e^{-x} + C \Rightarrow y = -x - 1 + C e^{x}$

//

$\int e^{-x} dx = \int e^{u} \cdot (-1) du = -\int e^{u} du = -e^{u} + C = -e^{-x} + C.$

$u = -x$

$du = -dx$

$(-1) du = dx$

Eq. de Bernoulli

$$y' + P(x)\,y = Q(x)\,y^n \quad , \quad n \neq 0 \ \text{e} \ n \neq 1$$

Aplicamos a mudança de variáveis $u = y^{1-n}$ e obtemos uma eq. linear em u.

Exemplo: $x y' + y = x^2 y^2 \overset{(\div x)}{\Rightarrow} y' + \frac{1}{x} y = x\, y^2$

$P = \frac{1}{x}$, $Q = x$, $n = 2$

$$u = y^{-1} = \frac{1}{y} \Rightarrow y = \frac{1}{u} \Rightarrow y^2 = \frac{1}{u^2}$$

$$y = u^{-1} \Rightarrow y' = (-1)\,u^{-2} \cdot u' \longrightarrow \left(\left([u(x)]^{-1} \right)' = (-1)\,[u(x)]^{-2} \cdot u'(x) \ \ \text{(regra da cadeia)} \right)$$

$$\therefore \ y' + \frac{1}{x} y = x\, y^2 \Rightarrow (-1)\,u^{-2} u' + \frac{1}{x} \cdot \frac{1}{u} = x \cdot \frac{1}{u^2}$$

$(x -u^2)$

$$\Rightarrow u' - \frac{1}{x} u = -x \quad \text{é linear}$$

Fator integrante: $e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

$$\therefore \left(u' - \frac{1}{x} u \right) \frac{1}{x} = -x \cdot \frac{1}{x} \Rightarrow \frac{1}{x} u' - \frac{1}{x^2} u = -1 \Rightarrow \left(\frac{1}{x} \cdot u \right)' = -1$$

$$\Rightarrow \int \left(\frac{1}{x} u \right)' dx = -\int 1\, dx \Rightarrow \frac{1}{x} u = -x + C \Rightarrow u = -x^2 + Cx$$

Portanto,

$$\frac{1}{y} = -x^2 + Cx \Rightarrow y = \frac{1}{-x^2 + Cx}.$$

(7) Use a mudança de variáveis $v = y/x$ para resolver a EDO $xy' = y + xe^{y/x}$.

$$v = \frac{y}{x} \Rightarrow y = x \cdot v(x) \Rightarrow y' = v + xv'$$

$$\therefore xy' = y + xe^{y/x} \Rightarrow x(v + xv') = xv + xe^{v}$$

$(\div x)$

$$\Rightarrow \cancel{v} + xv' = \cancel{v} + e^{v} \Rightarrow xv' = e^{v}$$

$$\Rightarrow e^{-v} \cdot v' = \frac{1}{x} \quad \text{(separável)}$$

$$\Rightarrow \int e^{-v}\, dv = \int \frac{1}{x}\, dx \Rightarrow -e^{-v} = \ln|x| + C$$

$$\Rightarrow e^{-v} = -\ln|x| - c \Rightarrow -v = \ln(-\ln|x| - c)$$

$$\Rightarrow v = -\ln(-\ln|x| - c)$$

Portanto,

$$\frac{y}{x} = -\ln(-\ln|x| - c) \Rightarrow y = -x\ln(-\ln|x| - c).$$